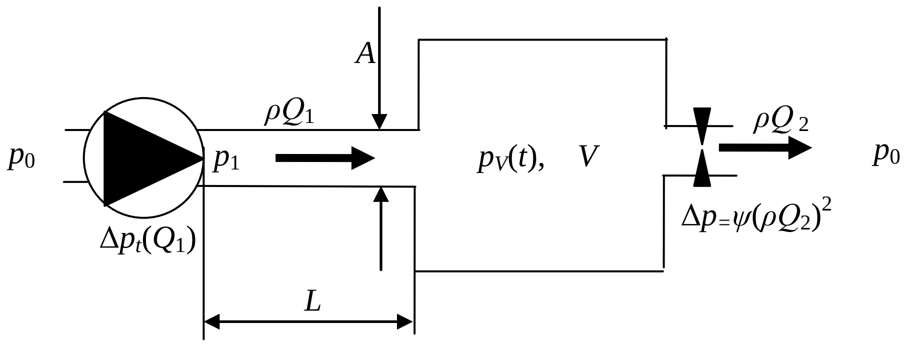
## Ventilation system

A ventilation system of a room consists of a fan, an air delivery pipe, the room to be ventilated and some local aerodynamic losses (gaps on windows, keyholes on doors). The sketch of such a system is shown below.




The fan feeds the air delivery pipe sucking air from the ambient atmosphere (stagnation pressure $p_0$) and raising the total pressure by $\Delta p_{tot}$. The total pressure rise depends on the volume flow rate ($Q_1$) or mass flow rate ($\dot{m}_1 = \rho\, Q_1$) passing through the fan. Thus we have

$$p_1 - p_0 = \Delta p_{tot} = f(\dot{m}_1) = F(Q_1) \tag{1}$$

The air pipe delivers the air according to the 1 dimensional unsteady Bernoulli equation where convective terms can be neglected.

$$\frac{L}{A}\frac{dQ_1}{dt} + \frac{p_V(t) - p_1}{\rho} = 0\,,$$

or after multiplying with the density

$$\frac{L}{A}\frac{d\dot{m}_1}{dt} + p_V(t) - p_1 = 0\,. \tag{2}$$

The volume $V$ of the room doesn't change. The air density in the room $\rho$ depends on the pressure $\rho(p_V)$. Thus the mass of air in the room is $m = \rho V$. The mass is changing in time if more air streams into the room than out of it:

$$\dot{m}_1 - \dot{m}_2 = \frac{dm}{dt} = \frac{d(\rho V)}{dt} = V\frac{d\rho}{dt} = V\frac{d\rho}{dp_V}\cdot\frac{dp_V}{dt}\,.$$

Here we have made use of the above mentioned barotropic $\rho(p_V)$ function. We know from Fluid Mechanics that the square of the sound velocity is

$$a^2 = \frac{dp_V}{d\rho}\,,$$

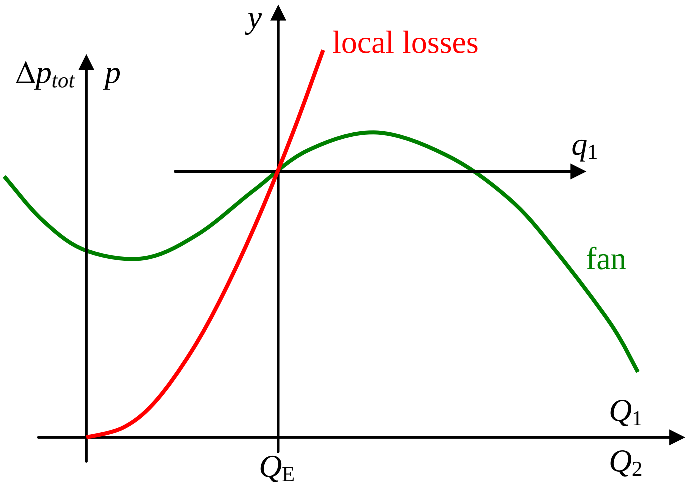
thus

$$\dot{m}_1 - \dot{m}_2 = V\frac{1}{a^2}\cdot\frac{dp_V}{dt}\,. \tag{3}$$

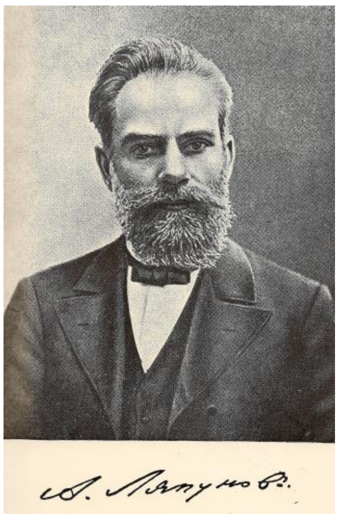
This is the equation of the room. Finally the local losses are described by supposing turbulent flow that is quadratic dependence of pressure drop on mass flow rate.

$$p_V(t) - p_0 = \psi\, \dot{m}_2^2 \,. \tag{4}$$

The steady state of the system can easily be found by putting the time derivatives equal zero: From Eq. (2) $p_{VE} - p_{1E} = 0$. Here the subscript $_E$ refers to equilibrium. Similarly from Eq. (3) $\dot{m}_{1E} - \dot{m}_{2E} = 0$. Thus there is one equilibrium mass flow rate denoted by $\dot{m}_{1E} = \dot{m}_{2E} = \dot{m}_E$ . From the fan's characteristics $p_{1E} = p_0 + \Delta p_{tot}(\dot{m}_E)$ and the local losses at steady state are $p_{VE} = p_0 + \psi\, \dot{m}_E^2 = p_{1E}$. These two equations together determine the equilibrium mass flow rate and room pressure: $p_0 + \psi \dot{m}_E^2 = p_{1E} = p_0 + \Delta p_{tot}(\dot{m}_E)$ which is an algebraic equation, graphically the intersection of the fan characteristics and the local loss characteristics gives the equilibrium mass flow rate.




An important question concerns the stability of the equilibrium of the ventilation system. There are different – linear and nonlinear – methods to investigate stability.



A nonlinear method originates from **A. M. Liapunov**.

The definition of the stability of an equilibrium point $\boldsymbol{x}=0$ of a system described by the ordinary differential equation (ODE) $\dot{\boldsymbol{x}}=\boldsymbol{\Phi}(\boldsymbol{x})$ is:

"The equilibrium state (trivial solution, $\boldsymbol{x}=0$) is stable if for any small value $\varepsilon$ there exists a value $\delta$ so that if the initial value of the solution of the ODE is closer to the equilibrium than $\delta$ than the solution $\boldsymbol{x}(t)$ remains in a region of radius $\varepsilon$ around the trivial solution." Here $\boldsymbol{x}(t)$ is a vector-scalar function, the dot denotes differentiation with respect to time $t$ and $\boldsymbol{\Phi}(\boldsymbol{x})$ is a nonlinear vector-vector function.

Liapunov's Theorem tells that the trivial solution of the system $\dot{\boldsymbol{x}}=\boldsymbol{\Phi}(\boldsymbol{x})$ is stable if there exists a scalar-vector Liapunov function $V(\boldsymbol{x},\dot{\boldsymbol{x}})\geq 0$ ($V = 0$ only for the trivial solution) with a negative semi definite time derivative $\dot{V}(\boldsymbol{x},\dot{\boldsymbol{x}})\leq 0$ according to the system.

If $\dot{V}(\boldsymbol{x},\dot{\boldsymbol{x}})<0$ – that is $\dot{V}$ is negative definite – then the trivial solution is also asymptotically stable.

The time derivative means the differential coefficient of $V$ considering the system $\dot{\boldsymbol{x}}=\boldsymbol{\Phi}(\boldsymbol{x})$.

$$\dot{V}(\boldsymbol{x},\dot{\boldsymbol{x}})=\left.\frac{dV}{dt}\right|_{\dot{\boldsymbol{x}}=\boldsymbol{\Phi}(\boldsymbol{x})}. \tag{5}$$

Thus starting from any initial solution close enough to the trivial solution the value of the positive definite Liapunov function is moving along an isolevel of $V$ or decreasing as $\dot{V}$ is negative semi definite or definite.

In order to apply Liapunov's theorem the above system of algebraic and differential equations must be transformed into a nonlinear 2$^{nd}$ order ODE.

Now we come back to our equation system. We transform the global system Eqs. (1)-(4) with unknowns $p_1$, $p_V$, $Q_1$, $Q_2$ to the local one: $y_1$, $y_V$, $q_1$, $q_2$ with following definitions:

$$y_1=\frac{p_1-p_{1E}}{\rho};\quad y_V=\frac{p_V-p_{1E}}{\rho};\quad q_1=Q_1-Q_E;\quad q_2=Q_2-Q_E. \tag{6}$$

Dividing Eq. (1) by $\rho$ and subtracting the equilibrium state of the same equation we have:

$$\frac{p_1}{\rho}-\frac{p_0}{\rho}=\frac{\Delta p_{tot}(Q_1)}{\rho}$$

$$\frac{p_{1E}}{\rho}-\frac{p_0}{\rho}=\frac{\Delta p_{tot}(Q_E)}{\rho}$$

$$\boxed{y_1=\frac{\Delta p_{tot}(Q_1)-\Delta p_{tot}(Q_E)}{\rho}=y(q_1).} \tag{7}$$

In a similar way from Eq (2) we have

$$\frac{L}{A}\frac{dQ_1}{dt}+\frac{p_V}{\rho}-\frac{p_1}{\rho}-\left[\frac{L}{A}\frac{dQ_E}{dt}+\frac{p_{1E}}{\rho}-\frac{p_{1E}}{\rho}\right]=\boxed{\frac{L}{A}\frac{dq_1}{dt}+y_V-y_1=0.} \tag{8}$$

From Eq. (3) follows

$$\boxed{q_1-q_2=\frac{V}{a^2}\frac{dy_V}{dt}.} \tag{9}$$

Finally by subtracting from Eq. (4) the equilibrium state of this equation an linearizing we have

$$\frac{p_V}{\rho}-\frac{p_0}{\rho}=\frac{\psi}{\rho}\rho^2 Q_2^2=\psi\rho Q_2^2$$
$$\frac{p_{VE}}{\rho}-\frac{p_0}{\rho}=\frac{p_{1E}}{\rho}-\frac{p_0}{\rho}=\psi\rho Q_E^2$$

$$\psi\rho\left(Q_2^2-Q_E^2\right)=\psi\rho\left(Q_E^2+2Q_E q_2+q_2^2-Q_E^2\right)\approx\psi\rho 2Q_E q_2=\boxed{kq_2=y_V.}\tag{10}$$

Now we combine Eqs. (7)-(10) and eliminate all dependent variables except $q_1$. From now on time derivatives are denoted by dots above letters. Multiply Eq. (8) by $L/A$ and differentiate it with respect to time $t$.

$\ddot{q}_1+\frac{A}{L}\left(\dot{y}_V-\dot{y}_1\right)=0$. From Eq (7) $\dot{y}_1=\frac{dy}{dq_1}\dot{q}_1$. From Eq. (9) $\dot{y}_V=\frac{a^2}{V}\left(q_1-q_2\right)$. Substitute the 2nd and 3rd into the 1st : $\ddot{q}_1+\frac{A}{L}\left(\frac{a^2}{V}\left(q_1-q_2\right)-\frac{dy}{dq_1}\dot{q}_1\right)=0$. Now only $q_2$ must be eliminated. But $q_2=\frac{y_V}{k}$ from (10), thus $q_2=\frac{1}{k}\left(y_1-\frac{L}{A}\dot{q}_1\right)=\frac{1}{k}\left(y(q_1)-\frac{L}{A}\dot{q}_1\right)$. Finally

$$\ddot{q}_1+\frac{A}{L}\left(\frac{a^2}{V}\left\{q_1-\left[\frac{1}{k}\left(y(q_1)-\frac{L}{A}\dot{q}_1\right)\right]\right\}-\frac{dy}{dq_1}\dot{q}_1\right)=0.$$

The terms must be written in the order of time derivatives:

$$\ddot{q}_1+\frac{A}{L}\left(\frac{a^2}{V}\frac{L}{A}\frac{1}{k}-\frac{dy}{dq_1}\right)\dot{q}_1+\frac{A}{L}\frac{a^2}{V}\left(q_1-\frac{1}{k}y(q_1)\right)=0.$$

The coefficients of the 1st and 0th derivative will be denoted by $\varphi(q_1)$ and $f(q_1)$ resulting in the general form of the 2nd order nonlinear differential equation:

$$\boxed{\ddot{q}_1+\varphi(q_1)\dot{q}_1+f(q_1)=0.}\tag{11}$$

A "good" Liapunov function for this type of equation is

$$\boxed{V(q_1,\dot{q}_1)=\frac{1}{2}\left(\dot{q}_1+\int_0^{q_1}\varphi(q)dq\right)^2+\int_0^{q_1}f(q)dq.}\tag{12}$$

The time derivative of $V$ considering Eq. (11) will be

$$\dot{V}(q_1,\dot{q}_1)=\frac{1}{2}2\left(\dot{q}_1+\int_0^{q_1}\varphi(q)dq\right)\cdot\left(\ddot{q}_1+\varphi(q_1)\cdot\dot{q}_1\right)+f(q_1)\cdot\dot{q}_1=$$
$$\dot{q}_1\underline{\left(\ddot{q}_1+\varphi(q_1)\cdot\dot{q}_1+f(q_1)\right)}+\left(\int_0^{q_1}\varphi(q)dq\right)\underline{\underline{\left(\ddot{q}_1+\varphi(q_1)\cdot\dot{q}_1\right)}}=\dot{q}_1\cdot\underline{0}+\left(\int_0^{q_1}\varphi(q)dq\right)\underline{\underline{\left(-f(q_1)\right)}}\Rightarrow$$

$$\boxed{\dot{V}(q_1,\dot{q}_1) = -f(q_1)\int_0^{q_1}\varphi(q)dq\,.} \tag{13}$$

The assumptions of Liapunov's theorem are surely fulfilled if the 2nd term in (12) is positive – the 1st term is quadratic, thus it is always non negative. This is true if

$f(q_1) = \frac{1}{k}\frac{A}{L}\frac{a^2}{V}\left(q_1 - \frac{1}{k}y(q_1)\right)q_1 > 0$. Because of (13) $\int_0^{q_1}\varphi(q)dq > 0$ must be fulfilled too.

From the first $q_1 > \frac{1}{k}y(q_1) > 0$ if $q_1 > 0$ or after rearrangement

$$\boxed{\frac{y(q_1)}{q_1} < k} \tag{14}$$

and the same must be true for $q_1 < 0$.

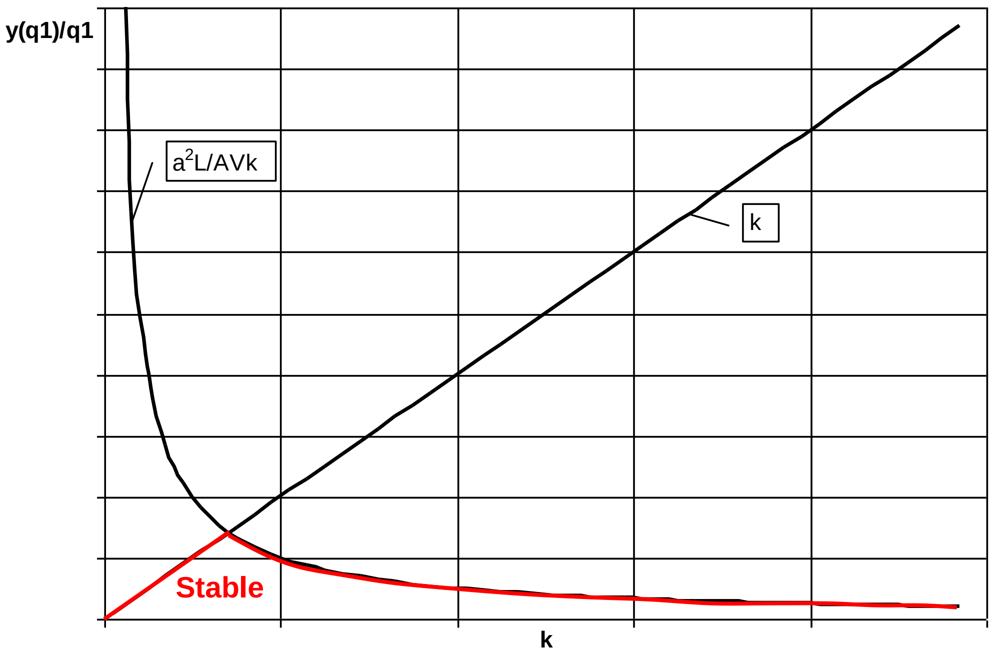
Coming back to the assumption

$$\int_0^{q_1}\varphi(q)dq = \int_0^{q_1}\frac{A}{L}\left(\frac{a^2}{V}\frac{L}{A}\frac{1}{k} - \frac{dy}{dq_1}\right)dq = \frac{A}{L}\left[\frac{a^2}{V}\frac{L}{A}\frac{1}{k}q_1 - \left(y(q_1) - y(0)\right)\right] = \frac{A}{L}\left[\frac{a^2}{V}\frac{L}{A}\frac{1}{k}q_1 - y(q_1)\right] > 0$$

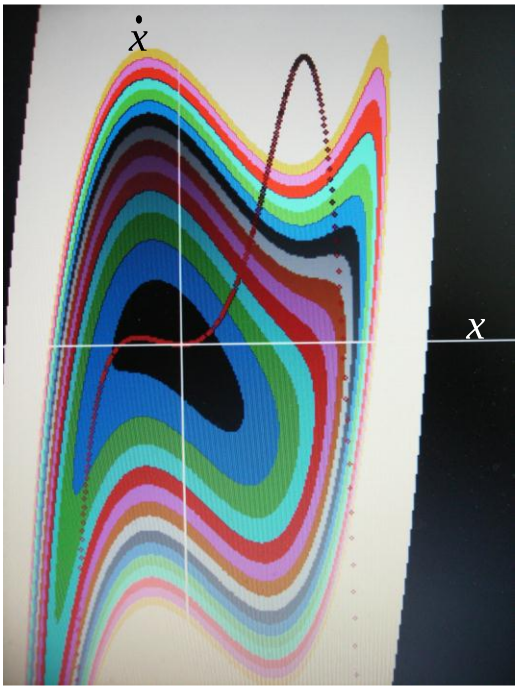
After dropping the positive $A/L$ and rearranging: $\frac{a^2L}{VA}\frac{1}{k}q_1 > y(q_1)$. This is fulfilled if

$$\boxed{\frac{y(q_1)}{q_1} < \frac{a^2L}{VA}\frac{1}{k}\,.} \tag{15}$$

The graphical interpretation of the stability criteria are shown in the figure below. The stable region is below the red lines.



The Liapunov function for one special case has the following form. Different colors mean different levels of the $V(\mathrm{x},\dot{\mathrm{x}})$ function. The thick red line is $\dot{V}(x,\dot{x})$ it depends in this case only on *x*.




**Amplitude amplification of the ventilation system**

The above described ventilation system is fed by a fan. The pressure at the pressure side of the fan is never constant, not even at steady state. We may suppose, that the actual pressure has the form $p_1 = p_E\left(1+\alpha\, e^{i\omega t}\right)$ where $\alpha$ is the amplitude of the perturbation of pressure and $\omega$ is its angular frequency. Physically the perturbation is nearly a real harmonic function but the calculation much easier if we suppose a complex harmonic function $e^{i\omega t}$. As we have prescribed the time variation of $p_1(t)$ only Eqs. (2)-(4) are needed to get the answer of the system:

$$\frac{L}{A}\frac{d\dot{m}_1}{dt} + p_V(t) - p_1 = 0 \tag{2}$$

$$\dot{m}_1 - \dot{m}_2 = V\frac{1}{a^2}\cdot\frac{dp_V}{dt} \tag{3}$$

$$p_V(t) - p_0 = \psi\,\dot{m}_2^2 \tag{4}$$

We differentiate (3) with respect to time and substitute the time derivatives of the mass flow rate from (2) and (4). To do the letter we find from (4): $\frac{dp_V}{dt} = \psi\, 2\dot{m}_2\frac{d\dot{m}_2}{dt} \approx \psi\, 2\dot{m}_E\frac{d\dot{m}_2}{dt}$

$$\frac{V}{a^2}\cdot\frac{d^2p_V}{dt^2} = \frac{d\dot{m}_1}{dt} - \frac{d\dot{m}_2}{dt} = \frac{A}{L}p_1(t) - \frac{A}{L}p_V - \frac{1}{2\psi\dot{m}_E}\cdot\frac{dp_V}{dt}\,.$$

We use again do dots for derivatives of the pressure in the room. The above equation reads after multiplying it with $L/A$:

$$\frac{VL}{a^2 A}\cdot \ddot{p}_V + \frac{L}{2\psi \dot{m}_E A}\cdot \dot{p}_V + p_V = p_1(t) = p_E\left(1+\alpha\, e^{i\omega t}\right). \tag{16}$$

One can look for the pressure in the room in a similar perturbed form as $p_1$ has been prescribed: $p_V = p_E\left(1+\beta\, e^{i\omega t}\right)$, thus the complex amplitude is different from $\alpha$ both in phase and magnitude. The absolute value of the ratio of $\beta$ to $\alpha$ is the amplification of the ventilation system and the phase shift can be calculated too. Differentiating $p_V = p_E\left(1+\beta\, e^{i\omega t}\right)$ and substituting into (16) gives

$$\frac{VL}{a^2 A}\cdot \underline{\underline{p_E}}\beta(i\omega)^2 \underline{\underline{e^{i\omega t}}} + \frac{L}{2\psi \dot{m}_E A}\cdot \underline{\underline{p_E}}\beta i\omega \underline{\underline{e^{i\omega t}}} + \underline{p_E} + \underline{\underline{p_E}}\beta \underline{\underline{e^{i\omega t}}} = \underline{p_E} + \underline{\underline{p_E}}\alpha\, \underline{\underline{e^{i\omega t}}}.$$

First drop the underlined terms then shorten by the twice underlined exponential function as it is never zero!

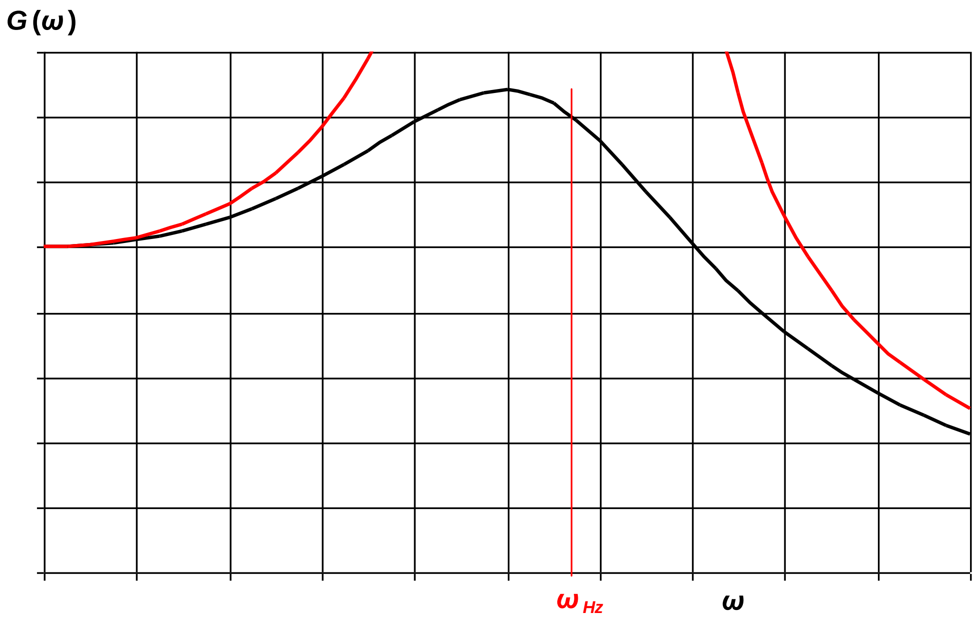
$$-\frac{VL}{a^2 A}\beta\omega^2 + \frac{L}{2\psi \dot{m}_E A}\beta\, i\omega + \beta = \alpha \;.$$

From here

$$\frac{\beta}{\alpha} = \frac{1}{1-\dfrac{VL}{a^2 A}\omega^2 + i\dfrac{\omega L}{2\psi \dot{m}_E A}} = \frac{1}{1-\dfrac{\omega^2}{\omega_H^2} + i\dfrac{\omega L}{2\psi \dot{m}_E A}}. \tag{17}$$

The Helmholtz-frequency $\omega_H = \sqrt{\dfrac{a^2 A}{VL}}$ has been introduced. The absolute value of the complex amplification is denoted by $G(\omega)$, it is

$$G(\omega) = \left|\frac{\beta}{\alpha}\right| = \frac{1}{\left|1-\dfrac{\omega^2}{\omega_H^2} + i\dfrac{\omega L}{2\psi \dot{m}_E A}\right|} = \frac{1}{\sqrt{\left(1-\dfrac{\omega^2}{\omega_H^2}\right)^2 + \left(\dfrac{\omega L}{2\psi \dot{m}_E A}\right)^2}} \tag{18}$$



The black line in the above figure is the graph of Eq. (18). The red line is the graph of a simple Helmholtz resonator – when the second term in the denominator in (18) is zero – with the Helmholtz resonator frequency as defined on the top of this page. If $\omega = \omega_H$ then the amplification of an undamped resonator goes to infinity.

The phase angle between excitation and the answer of the system is

$$\varphi(\omega) = \mathrm{a}\tan\left(\frac{\dfrac{\omega L}{2\psi \dot{m}_E A}}{1-\dfrac{\omega^2}{\omega_H^2}}\right). \tag{19}$$

This is shown in the next figure, a phase jump of about 180° is seen at the Helmholtz frequency.